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Quantum Stochastic Processes with Independent Additive Increments

MICHAEL SCHÜRMANN

Institut für Angewandte Mathematik, Universität Heidelberg, Im Neuenheimer Feld 249, D-6900 Heidelberg 1, Germany

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We give a complete characterization of a class of quantum stochastic processes with independent, stationary increments. We prove that processes of the class are, up to a canonical equivalence, equal to a sum of creation, second quantization, annihilation, and scalar processes on a Bose/Fermi Fock space, showing that, with our notion of independence, there are no other "white noises" but those used in the quantum stochastic calculus of R. L. Hudson and K. R. Parthasarathy.

1. INTRODUCTION

One of the basic theorems of classical probability theory is the classification of all stochastic processes with independent and stationary increments taking values in a d-dimensional euclidean space and their construction from the building blocks of Brownian motion and Poisson process. We show that a similar result holds for quantum stochastic processes under the restriction that all moments exist. The processes classified in this paper are the quantum stochastic processes with independent, stationary additive increments of [3].

The representations of our processes will live on Bose/Fermi Fock space. Fock space plays also an important role in classical probability theory; see [18]. It is well known that Brownian motion and the Poisson process can be realized as operator processes on Bose Fock space. The space $L^2(\mathscr{W})$ of square-integrable functions on Wiener space $\mathscr{W}$ can be identified with the Bose Fock space over $L^2(\mathbb{R}_+)$ by the Wiener chaos representation theorem. Then Brownian motion gives rise to a linear operator on $L^2(\mathscr{W})$

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by pointwise multiplication. On Fock space this operator is represented by the sum $a_t^\dagger + a_t$ of the creation process $a_t^\dagger = a^\dagger(\chi_{(0,t)})$ and the annihilation process $a_t = a(\chi_{(0,t)})$, where $\chi_{(0,t)} \in L^2(\mathbb{R}_+)$ denotes the characteristic function of the interval $(0, t)$ in $\mathbb{R}_+$. Moreover, the probability law of the self-adjoint operators $a_t^\dagger + a_t$ in the vacuum state is that of standard Brownian motion. Let λ_t be the differential second quantization of the operator on $L^2(\mathbb{R}_+)$ obtained from multiplication by $\chi_{(0,t)}$. If we add λ_t to Brownian motion we obtain self-adjoint operators $a_t^\dagger + \lambda_t + a_t$ on Fock space whose probability law in the vacuum state is that of a centralized Poisson process of intensity 1; see [15].

If we look at the (non-commuting) creation and annihilation processes as a pair $(a_t^\dagger, a_t)$, then this pair forms a quantum Wiener process in the sense of [7]. Similarly, multivariate quantum Wiener processes are vectors of pairs of creation and annihilation processes on the Fock space over $L^2(\mathbb{R}_+) \otimes \mathbb{R}^d$. The quantum stochastic integral against creation, differential second quantization (or preservation) and annihilation processes on Fock space was developed by R. L. Hudson and K. R. Parthasarathy [15] for the Bose case and by D. Applebaum and R. L. Hudson [5] for the Fermi case (see also [16] for a unification of these theories). It generalizes the classical theory of Ito stochastic integration against Brownian motion and the Poisson process. Quantum probability can throw a new light on classical probability and provide new techniques; cf. [18]. For example, M. Emery [9] found a class of martingales which have the chaos completeness property but are not classical stochastic processes with independent stationary increments, and K. R. Parthasarathy [20] showed that these martingales are solutions of certain quantum stochastic differential equations. He exploited this to give a completely different proof of Emery's result. Using quantum stochastic calculus, more new and interesting classical stochastic processes may be found.

This leads to the important question as to whether there is some arbitrariness in the choice of integrators in quantum stochastic calculus. Are there good reasons to consider precisely the above mentioned processes on Fock space? It seems natural to accept the quantum stochastic processes with independent, stationary additive increments of [3] as candidates for a non-commutative stochastic integration theory. The purpose of this paper is to give a characterization of these processes. We prove that they are, up to a canonical equivalence, equal to sums of creation, preservation, annihilation, and scalar processes on a Bose/Fermi Fock space over $L^2(\mathbb{R}_+) \otimes H$ with H an appropriate Hilbert space. So, with our notion of independence, we see that, in a sense, Bose/Fermi quantum stochastic calculus covers all possible theories of non-commutative stochastic integration against "white noise."

In view of classical stochastic integration theory, one may also look for a characterization of quantum martingales. The quantum martingale representation theorems of [14, 21, 22], however, *start* from processes defined on Fock space over $L^2(\mathbb{R}_+)$, whereas in the representation theorem of the present paper one of the main points is that the Fock space structure *results* from properties of the processes. The "quantum Lévy characterization" of quantum Brownian motion (see [1, 19]) goes more in the direction of the present paper. In [1, 19] our condition of independence of increments is partly replaced by a weaker martingale condition. On the other hand, the class of processes to be characterized is much smaller, and this is why there is an additional condition on the process replacing P. Lévy's condition of continuity of the trajectories.

It should be mentioned that there are other notions of non-commutative independence than the one of this paper. Recently, free (or Voiculescu) independence was used to develop a quantum stochastic calculus on the full (not symmetrized) Fock space; see [26, 27]. This and other choices of independence will be subject to further investigations.

In this paper we make no use of the result of [24] where a formula for the Gelfand–Naimark–Segal (GNS) representation of an infinitely divisible state on a cocommutative *-bialgebra was developed. Looking from a different point of view, we here work directly with the quantum stochastic processes.

Using arguments similar to those applied in this paper, a representation theorem for non-commutative *unitary* processes with independent and stationary (multiplicative) increments was proved in [23]. The case of bounded processes with independent and stationary increments was treated in [11].

The paper is organized as follows. In Section 2 we give the basic definitions and review results from [3] needed for the proof of the representation theorem. The first part of Section 3 introduces the "mixed" Bose/Fermi Fock space and the operators which form the building blocks of our processes. Then we prove the representation theorem. Finally, in Section 4 examples like the quantum Wiener process and the quantum Poisson process [15, 25] are discussed, and it is shown that a given quantum stochastic process with independent, stationary additive increments has a "maximal quantum Wiener component."

2. Basic Concepts

Quantum stochastic processes are generalizations of classical stochastic processes in the sense that algebras of scalar valued functions are replaced by not necessarily commutative *-algebras; cf. [2]. We consider processes with finite moments of all orders, so that the state space of the process can be represented by an algebra of polynomial functions. Following the ideas of [10, 12, 28], non-commutativity is then introduced by passing to polynomial algebras in *non-commuting* indeterminates. We explain this in more detail. Let $(X_t)_{t\in\mathbb{R}_+}$ be a classical stochastic process taking values in $\mathbb{R}^d$, $d\in\mathbb{N}$, defined on some probability space $(E, \mathscr{E}, P)$. For $\omega\in E$ we write $X_t(\omega)$ as a vector $(X_t^{(1)}(\omega), ..., X_t^{(d)}(\omega))$. We assume that all the moments

$$m(t; l_1, ..., l_d)=\int_E (X_t^{(1)})^{l_1}\cdots(X_t^{(d)})^{l_d}\,dP,$$

$t\in\mathbb{R}_+$, $l_1, ..., l_d\in\mathbb{N}\cup\{0\}$, of the process exist, that the increments $X_{st}=X_t-X_s$, $s\leqslant t$, are independent for disjoint intervals, and that the distribution of X_{st} only depends on the difference $t-s$ (stationarity of increments). Examples are Brownian motion and the Poisson process. Since we assume that all the moments of the distribution P_t of X_t exist, P_t gives rise to a state φ_t on the *-algebra $\mathbb{C}[d]$ of polynomials with complex coefficients in d real commuting indeterminates $x_1, ..., x_d$. It is well known that, in passing from P_t to φ_t, we loose information (classical moment problem), but in many applications the moments of the process are all one is interested in. The states φ_t form a one-parameter convolution semi-group and they determine the moments $m(t; l_1, ..., l_d)$. The complex, unital algebra $\mathscr{D}$ of functions on E generated by the constant functions and by the components $X_t^{(l)}$, $t\in\mathbb{R}_+$, $l\in\{1, ..., d\}$, of the process is considered as a linear subspace of the Hilbert space $L^2(E, \mathscr{E}, P)$. Denote by $F_t^{(l)}$ the symmetric operator on the pre-Hilbert space $\mathscr{D}$ which is the multiplication by $X_t^{(l)}$. As an element of $L^2(E, \mathscr{E}, P)$ the constant function $\mathbf{1}$ is denoted by Φ. We have

$$\varphi_t(x_1^{l_1}\cdots x_d^{l_d})=m(t; l_1, ..., l_d)=\langle\Phi, (F_t^{(1)})^{l_1}\cdots(F_t^{(d)})^{l_d}\Phi\rangle.$$

We are now in the following situation:

(a) $\mathscr{D}$ is a pre-Hilbert space.

(b) Φ is a unit vector in $\mathscr{D}$.

(c) $(F_t)_{t\in\mathbb{R}_+}$ is a family of d-tuples $F_t=(F_t^{(1)}, ..., F_t^{(d)})$ of symmetric operators on $\mathscr{D}$.

We denote by $\mathscr{A}_{st}$, $s\leqslant t$, the complex, unital algebra of linear operators on $\mathscr{D}$ generated by the identity and by the additive increments $F_{st}^{(l)}=F_t^{(l)}-F_s^{(l)}$. Then independence of increments reads

(d)

$$\langle\Phi, A_1\cdots A_m\Phi\rangle=\langle\Phi, A_1\Phi\rangle\cdots\langle\Phi, A_m\Phi\rangle \tag{2.1}$$

for $m\in\mathbb{N}$ and $t_1<t_2<\cdots<t_{m+1}$, $A_l\in\mathscr{A}_{t_l,t_{l+1}}$.

Stationarity of increments becomes

(e) The distribution φ_{st} of F_{st} only depends on the difference $t-s$.

Here "distribution" means the restriction to $\mathbb{C}[d]$ of the distribution of the corresponding random variable.

Now we extend the framework of classical probability theory and *start* from a general situation like that of (a)–(e). But what is a distribution? The distribution of a d-tuple $F=(F^{(1)}, ..., F^{(d)})$ of symmetric operators on $\mathscr{D}$ is now the state φ_F on the *-algebra $\mathbb{C}\langle d\rangle$ of polynomials in *d non-commuting* indeterminates $x_1, ..., x_d$ given by

$$\varphi_F(x_{i_1} \cdots x_{i_m}) = \langle \Phi, F^{(i_1)} \cdots F^{(i_m)} \Phi \rangle,$$

$m \in \mathbb{N}$, $i_1, ..., i_m \in \{1, ..., d\}$. The distribution φ_F is again the "moment functional" of F, but since the components of F are not assumed to commute, we have to take into account the order of the x_i. As far as (d) is concerned we make an important additional assumption which is trivial in the commutative case. We assume that the algebras $\mathscr{A}_{st}$ and $\mathscr{A}_{s't'}$ commute for disjoint intervals (s, t) and (s', t') of $\mathbb{R}_+$. Notice that this does not mean reduction to the commutative case, since the algebras $\mathscr{A}_{st}$ themselves are allowed to be non-commutative. The non-commutative notion of independence considered here was proposed in [8] and might be called *Bose/Fermi independence.* In the sequel, families $(F_t)_{t \in \mathbb{R}_+}$ of operators with the discussed properties are called "(quantum) stochastic processes with independent and stationary increments." (In [3] this term was used for a much wider class of processes, and the processes considered in this paper were called "processes with independent, stationary *additive* increments.")

Before we state our result on the representation of processes in the next section we introduce a slightly more general setting. This is done to include *graded* algebras which enables us to treat the Fermionic case.

Let $V = V^{(0)} \oplus V^{(1)}$ be a graded complex vector space. A *conjugation* on V is an antilinear, even mapping $v \mapsto v^*$ on V such that $(v^*)^* = v$. For example, take V to be the complex vector space spanned by $\{x_1, ..., x_d\}$ with the trivial graduation and the conjugation given by extension of $(x_i)^* = x_i$. In fact, the general case is obtained by starting from a family $\{x_i\}_{i \in I}$ of hermitian indeterminates, dividing them into even and odd indeterminates and taking V to be the complex vector space with basis $\{x_i\}$. We denote by $\deg v$ the degree of v if v is homogeneous. We denote by $\mathscr{T}(V)$ the graded tensor algebra over V, that is

$$\mathscr{T}(V) = \mathbb{C}\mathbf{1} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

and multiplication in $\mathcal{T}(V)$ is given by

$$(v_1 \otimes \cdots \otimes v_m) w = v_1 \otimes \cdots \otimes v_m \otimes w,$$

$v_1, ..., v_m, w \in V$. The conjugation on V gives rise to an involution on $\mathcal{T}(V)$ which we again denote by *. Let $\mathcal{D}$ be a graded pre-Hilbert space and denote by $\mathbf{H}(\mathcal{D})$ the *-algebra of all linear operators on $\mathcal{D}$ with the property that the restriction of their adjoint is also a linear operator on $\mathcal{D}$. Let Φ be an even unit vector in $\mathcal{D}$. A *random variable* on $(\mathcal{D}, \Phi)$ is an even, hermitian linear mapping F from V to $\mathbf{H}(\mathcal{D})$. The universal property of $\mathcal{T}(V)$ yields that F can be extended to a *-algebra homomorphism j_F from $\mathcal{T}(V)$ to $\mathbf{H}(\mathcal{D})$ in a unique way. The *distribution* of F is the even state φ_F on $\mathcal{T}(V)$ given by

$$\varphi_F(a) = \langle \Phi, j_F(a) \Phi \rangle,$$

$a \in \mathcal{T}(V)$. The graded sub-*-algebras $\mathcal{A}_1, ..., \mathcal{A}_m$ of $\mathbf{H}(\mathcal{D})$ are called *independent* if for $A_l \in \mathcal{A}_l$ Eq. (2.1) holds and

$$[A_k, A_l] = 0 \qquad \text{for} \quad k \neq l,$$

where

$$[A, B] = AB - (-1)^{\deg A \deg B} BA$$

for (homogeneous) A, B in $\mathbf{H}(\mathcal{D})$. The random variables $F_1, ..., F_m$ are called independent if the *-algebras $j_{F_1}(\mathcal{T}(V)), ..., j_{F_m}(\mathcal{T}(V))$ are independent. We say that a net $(F_\alpha)_{\alpha \in A}$ of random variables *converges* to a random variable F *in distribution* if $(\varphi_{F_\alpha}(a))_{\alpha \in A}$ converges to $\varphi_F(a)$ for all a in $\mathcal{T}(V)$. A *quantum stochastic process* on $(\mathcal{D}, \Phi)$ in the sense of this paper is a family $\mathcal{F} = (F_t)_{t \in \mathbb{R}_+}$ of random variables on $(\mathcal{D}, \Phi)$. A quantum stochastic process is called *minimal* if $\mathcal{D}$ is equal to the linear span of $\{\Phi\} \cup \{F_{t_1}(v_1) \cdots F_{t_m}(v_m)\Phi \colon m \in \mathbb{N},\ t_1, ..., t_m \in \mathbb{R}_+, v_1, ..., v_m \in V\}$. Two quantum stochastic processes $(F_{i,t})_{t \in \mathbb{R}_+}$ on $(\mathcal{D}_i, \Phi_i)$, $i = 1, 2$, are called *stochastically equivalent* if for all $m \in \mathbb{N}$, $t_1, ..., t_m \in \mathbb{R}_+$, and $a_1, ..., a_m \in \mathcal{T}(V)$ the numbers

$$\langle \Phi_i, j_{i,t_1}(a_1) \cdots j_{i,t_m}(a_m) \Phi_i \rangle$$

are the same for $i = 1, 2$ (where we set $j_{F_{i,t}} = j_{i,t}$). If both processes are minimal this is equivalent to the following. There is a unitary operator $\mathcal{U}$ from the completion of $\mathcal{D}_1$ to the completion of $\mathcal{D}_2$ such that $\mathcal{U}\Phi_1 = \Phi_2$ and $\mathcal{U}F_{1,t}(v) = F_{2,t}(v)\mathcal{U}$ for all $v \in V$.

We always make the assumption that F_t converges to 0 in distribution for $t \downarrow 0$.

The *increments* of the process are the random variables F_{st}, $s \leqslant t$. We say that $\mathcal{F}$ has *independent increments* if for all $n \in \mathbb{N}$ and $t_1, ..., t_{n+1} \in \mathbb{R}_+$, $t_1 < t_2 < \cdots < t_{n+1}$, the increments $F_{t_1 t_2}, ..., F_{t_n t_{n+1}}$ are independent. We say that $\mathcal{F}$ has *stationary increments* if the distribution of F_{st} only depends on $t-s$.

We give a brief review of some results of [3]. It is well known that classical stochastic processes with independent, stationary increments are determined, up to stochastic equivalence, by the one-parameter convolution semi-group $\{P_t : t \in \mathbb{R}_+\}$ of distributions P_t of X_t. Moreover, if the process is weakly continuous it can be characterized through its exponent function, that is, the logarithm of the Fourier transform of the distribution of X_1 (Lévy–Khintchine formula). Of course, one may as well consider the semi-group of states on the *-algebra of bounded continuous functions on $\mathbb{R}^d$ given by $\{P_t\}$. For the general case of a Lie group valued stochastic process, the generator of this semi-group is described by Hunt's formula [17]. The convolution product of measures on $\mathbb{R}^d$ becomes the following product of linear functionals on $\mathbb{C}[d]$. Define the homomorphism

$$\Delta : \mathbb{C}[d] \to \mathbb{C}[d] \otimes \mathbb{C}[d]$$

by fixing its value on a generator x_i of $\mathbb{C}[d]$ to be $x_i \otimes \mathbf{1} + \mathbf{1} \otimes x_i$. The convolution product of two linear functionals ψ and φ on $\mathbb{C}[d]$ is then given by

$$\psi * \varphi = (\psi \otimes \varphi) \circ \Delta. \tag{2.2}$$

A convolution product of linear functionals on $\mathcal{T}(V)$ is defined exactly as for $\mathbb{C}[d]$ by Eq. (2.2), only that now the mapping Δ denotes the even *-algebra homomorphism from $\mathcal{T}(V)$ to $\mathcal{T}(V) \otimes \mathcal{T}(V)$ with $\Delta(v) = v \otimes \mathbf{1} + \mathbf{1} \otimes v$ for $v \in V$. Here the *-algebra structure of $\mathcal{T}(V) \otimes \mathcal{T}(V)$ is given by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg b \deg a'} aa' \otimes bb'$$

and

$$(a \otimes b)^* = (-1)^{\deg a \deg b} a^* \otimes b^*,$$

a, b, a', $b' \in \mathcal{T}(V)$. It is immediate to check that the distribution δ of 0 is the unit of the convolution product (2.2). The *-algebra $\mathcal{T}(V)$ is an example of a graded *-bialgebra with comultiplication Δ and counit δ; see [3]. For two independent random variables F and G we have

$$\varphi_{F+G} = \varphi_F * \varphi_G.$$

Therefore, for a quantum stochastic process $\mathscr{F}$ with independent and stationary increments the distributions φ_t of F_t form a one-parameter convolution semi-group of states on $\mathscr{T}(V)$. Moreover, as a consequence of the continuity of φ_t at the origin the functions $t \mapsto \varphi_t(a)$ are differentiable at the origin for all $a \in \mathscr{T}(V)$. The linear functional ψ on $\mathscr{T}(V)$ given by

$$\psi(a) = \frac{d}{dt} \varphi_t(a) \bigg|_{t=0}$$

vanishes at $\mathbf{1}$ and is even, hermitian, and conditionally positive, where "conditionally positive" means

$$\psi(a^* a) \geqslant 0 \qquad \text{for all} \quad a \in \mathscr{T}(V) \text{ with } \delta(a) = 0.$$

In the following "conditionally positive" will always include $\psi(\mathbf{1}) = 0$. The distribution φ_t is the convolution exponential

$$\exp_*(t\psi) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \psi^{*n},$$

where ψ^{*n} denotes the nth convolution power of ψ and the limit on the right-hand side is to be understood pointwise. The linear functional ψ is called the *generator* of the process. In fact, two processes with independent and stationary increments are stochastically equivalent if and only if their generators coincide. On the other hand, for a given even, hermitian, conditionally positive linear functional ψ their exists a process $\mathscr{F}$ with independent and stationary increments such that ψ is the generator of $\mathscr{F}$. Thus, there is a 1–1 correspondence between even, hermitian, conditionally positive linear functionals on $\mathscr{T}(V)$ and (equivalence classes of) processes with independent and stationary increments.

3. Representation Theorem

We begin this section with the construction of a class of processes with independent and stationary increments. Then we prove that each process with independent and stationary increments is stochastically equivalent to a process in this class.

For graded vector spaces V and W and linear operators A and B on V and W, respectively, we denote by $A \otimes B$ the linear operator on the graded vector space $V \otimes W$ defined by

$$(A \otimes B)(v \otimes w) = (-1)^{\deg B \deg v} A(v) \otimes B(w).$$

Let D be a graded pre-Hilbert space. We form the "graded tensor pre-Hilbert space"

$$\mathscr{T}(D)=\mathbb{C}\oplus D\oplus(D\otimes D)\oplus(D\otimes D\otimes D)\oplus\cdots$$

of D. The vector $1\oplus 0\oplus 0\oplus\cdots$ in $\mathscr{T}(D)$ is called the *vacuum vector* and is denoted by Ω. The symmetric group S_n acts on the n-fold tensor product $D^{\otimes n}$ of D with itself through

$$\pi(\xi_1\otimes\cdots\otimes\xi_m)=\operatorname{sgn}(\pi;\varepsilon_1,...,\varepsilon_m)\,\xi_{\pi^{-1}(1)}\otimes\cdots\otimes\xi_{\pi^{-1}(m)},$$

where ξ_l are homogeneous elements of D and for $\varepsilon\in\mathbb{Z}_2$ the factor $\operatorname{sgn}(\pi;\varepsilon_1,...,\varepsilon_m)$ denotes the sign of the permutation of $\#\{l:\varepsilon_l=1\}$ elements derived from

$$\pi=\begin{pmatrix}1&\cdots&n\\ \pi(1)&\cdots&\pi(n)\end{pmatrix}$$

by eliminating the numbers l in the upper and lower row for which $\varepsilon_l=0$. We then define the linear operator $\mathscr{P}_D$ on $\mathscr{T}(D)$ by

$$\mathscr{P}_D(\xi_1\otimes\cdots\otimes\xi_m)=\frac{1}{m!}\sum_{\pi\in S_m}\pi(\xi_1\otimes\cdots\otimes\xi_m).$$

We define the linear operator

$$\tilde{\Theta}:\mathscr{T}(D)\to\mathscr{T}(D^{(0)})\otimes\mathscr{T}(D^{(1)})$$

by

$$\tilde{\Theta}(\xi_1\otimes\cdots\otimes\xi_m)=\binom{m}{k}^{-1/2}\zeta_1\otimes\cdots\otimes\zeta_k\otimes\zeta_{k+1}\otimes\cdots\otimes\zeta_m,$$

where $\zeta_1,...,\zeta_k$ are the even and $\zeta_{k+1},...,\zeta_m$ are the odd factors of the tensor $\xi_1\otimes\cdots\otimes\xi_m$. Moreover, define

$$\tilde{\Theta}^\dagger:\mathscr{T}(D^{(0)})\otimes\mathscr{T}(D^{(1)})\to\mathscr{T}(D)$$

by

$$\begin{aligned}&\tilde{\Theta}^\dagger(\zeta_1\otimes\cdots\otimes\zeta_k\otimes\zeta_{k+1}\otimes\cdots\otimes\zeta_m)\\ &\quad=\sum_{\pi\in S_{m,k}}\binom{m}{k}^{-1/2}\pi(\zeta_1\otimes\cdots\otimes\zeta_k\otimes\zeta_{k+1}\otimes\cdots\otimes\zeta_m),\end{aligned}$$

where $S_{m,k}$ denotes the subgroup of S_m formed by the permutations which respect the order of $\{1, ..., k\}$ and $\{k+1, ..., m\}$. Then

$$(\tilde{\Theta})^* = \tilde{\Theta}^\dagger$$
$$\tilde{\Theta} \circ (\tilde{\Theta})^* = \mathrm{id}$$
$$\mathscr{P}_D = (\tilde{\Theta})^* \circ (\mathscr{P}_{D^{(0)}} \otimes \mathscr{P}_{D^{(1)}}) \circ \tilde{\Theta}$$
$$(\mathscr{P}_D)^2 = \mathscr{P}_D$$
$$(\mathscr{P}_D)^* = \mathscr{P}_D$$
$$\tilde{\Theta} \circ \mathscr{P}_D = (\mathscr{P}_{D^{(0)}} \otimes \mathscr{P}_{D^{(1)}}) \circ \tilde{\Theta}.$$

We set $\mathscr{S}(D) = \mathscr{P}_D(\mathscr{T}(D))$. The restriction Θ of $\tilde{\Theta}$ to $\mathscr{S}(D)$ is an isomorphism from the pre-Hilbert space $\mathscr{S}(D)$ to the pre-Hilbert space $\mathscr{S}(D^{(0)}) \otimes \mathscr{S}(D^{(1)})$. If D is the orthogonal sum $D_1 \oplus D_2$ of two graded linear subspaces D_1 and D_2 we have

$$\mathscr{S}(D) \cong \mathscr{S}(D_1) \otimes \mathscr{S}(D_2), \tag{3.1}$$

where we use the canonical isomorphisms

$$\mathscr{S}(D^{(0)}) \cong \mathscr{S}(D_1^{(0)}) \otimes \mathscr{S}(D_2^{(0)})$$
$$\mathscr{S}(D^{(1)}) \cong \mathscr{S}(D_1^{(1)}) \otimes \mathscr{S}(D_2^{(1)});$$

see [5, 15].

For $\xi \in D$ define the operators $\tilde{a}^\dagger(\xi)$ and $\tilde{a}(\xi)$ on $\mathscr{T}(D)$ by

$$\tilde{a}^\dagger(\xi)\xi_1 \otimes \cdots \otimes \xi_n = (n+1)^{1/2}\, \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n$$
$$\tilde{a}^\dagger(\xi)\Omega = \xi$$

and

$$\tilde{a}(\xi)\xi_1 \otimes \cdots \otimes \xi_{n+1} = (n+1)^{1/2}\, \langle \xi, \xi_1 \rangle \xi_2 \otimes \cdots \otimes \xi_{n+1}$$
$$\tilde{a}(\xi)\zeta = \langle \xi, \zeta \rangle \Omega$$
$$\tilde{a}(\xi)\Omega = 0.$$

Then $\tilde{a}^\dagger(\xi) = \tilde{a}(\xi)^*$, and we define the *creation operators* $a^\dagger(\xi)$ and the *annihilation operators* $a(\xi)$ on $\mathscr{S}(D)$ by

$$a^\dagger(\xi) = \mathscr{P}_D \circ \tilde{a}^\dagger(\xi) \restriction \mathscr{S}(D)$$

and

$$a(\xi) = \tilde{a}(\xi) \restriction \mathscr{S}(D).$$

For $A\in\mathbf{H}(D)$ we define the operator $\tilde{\lambda}(A)$ on $\mathscr{T}(D)$ by

$$\tilde{\lambda}(A)\restriction D^{\otimes n}=A\otimes\mathrm{id}\cdots\otimes\mathrm{id}+\mathrm{id}\otimes A\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}+\cdots+\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes A$$

$$\tilde{\lambda}(A)\Omega=0.$$

The operators $\lambda(A)=\tilde{\lambda}(A)\restriction\mathscr{S}(D)$ are called *differential second quantization* or *preservation operators.* The mappings $\xi\mapsto a^\dagger(\xi)$ and $A\mapsto\lambda(A)$ are even, linear mappings from D to $\mathbf{H}(\mathscr{S}(D))$ and from $\mathbf{H}(D)$ to $\mathbf{H}(\mathscr{S}(D))$, respectively. The latter is also hermitian. The mapping $\xi\mapsto a(\xi)$ is an even, antilinear mapping from D to $\mathbf{H}(\mathscr{S}(D))$. For a decomposition $D=D_1\oplus D_2$ the canonical isomorphism between $\mathscr{S}(D)$ and $\mathscr{S}(D_1)\otimes\mathscr{S}(D_2)$ gives

$$a^\dagger(\xi_1\oplus\xi_2)\cong a^\dagger(\xi_1)\otimes\mathrm{id}+\mathrm{id}\otimes a^\dagger(\xi_2)\tag{3.2}$$

$$a(\xi_1\oplus\xi_2)\cong a(\xi_1)\otimes\mathrm{id}+\mathrm{id}\otimes a(\xi_2)\tag{3.3}$$

$$\lambda(A_1\oplus A_2)\cong\lambda(A_1)\otimes\mathrm{id}+\mathrm{id}\otimes\lambda(A_2)\tag{3.4}$$

for $\xi_1\in D_1$, $\xi_2\in D_2$, $A_1\in\mathbf{H}(D_1)$, and $A_2\in\mathbf{H}(D_2)$. Especially, for $D_1=D^{(0)}$ and $D_2=D^{(1)}$ we have

$$a^\dagger(\xi)\cong a^\dagger(\xi^{(0)})\otimes\mathrm{id}+\mathrm{id}\otimes a^\dagger(\xi^{(1)})$$

$$a(\xi)\cong a(\xi^{(0)})\otimes\mathrm{id}+\mathrm{id}\otimes a(\xi^{(1)}),$$

where $\xi\in D$ and $\xi^{(0)}$ and $\xi^{(1)}$ denote the even and the odd part of ξ. Notice that for an *odd* operator A in $\mathbf{H}(D)$ the operator $\lambda(A)$ does not leave invariant $\mathscr{S}(D^{(0)})$ or $\mathscr{S}(D^{(1)})$. We have the following commutation relations

$$[a(\xi),a(\zeta)]=[a^\dagger(\xi),a^\dagger(\zeta)]=0\tag{3.5}$$

$$[a(\xi),a^\dagger(\zeta)]=\langle\xi,\zeta\rangle\mathrm{id}\tag{3.6}$$

$$[\lambda(A),\lambda(B)]=\lambda([A,B])\tag{3.7}$$

$$[\lambda(A),a^\dagger(\xi)]=a^\dagger(A\xi)\tag{3.8}$$

$$[\lambda(A),a(\xi)]=-a(A^*\xi)\tag{3.9}$$

for $\xi,\zeta\in D$ and $A,B\in\mathbf{H}(D)$. We equip the Hilbert space $L^2(\mathbb{R}_+)$ with the trivial graduation. For $\xi\in D$ and $A\in\mathbf{H}(D)$ we define operators on $\mathscr{S}(L^2(\mathbb{R}_+)\otimes D)$ by

$$a_t^\dagger(\xi)=a^\dagger(\chi_{(0,t)}\otimes\xi)\tag{3.10}$$

$$a_t(\xi)=a(\chi_{(0,t)}\otimes\xi)\tag{3.11}$$

$$\lambda_t(A)=\lambda(\chi_{(0,t)}\otimes A),\tag{3.12}$$

where, in the third definition, we identified $\chi_{(0,t)} \in L^2(\mathbb{R}_+)$ with the corresponding multiplication operator on $L^2(\mathbb{R}_+)$. The three types of processes (3.10), (3.11), and (3.12) together with the "scalar processes" $(\alpha t \,\mathrm{id})_{t\in\mathbb{R}_+}$, $\alpha\in\mathbb{C}$, are the building blocks of processes with independent and stationary increments.

A *generator* Γ over a graded vector space V with conjugation is a quadruple $\Gamma=(D, \eta_0, \rho_0, \psi_0)$ consisting of

— a graded pre-Hilbert space D

— an even linear mapping $\eta_0\colon V\to D$

— an even, hermitian linear mapping $\rho_0\colon V\to \mathbf{H}(D)$ such that $D=\{j_{\rho_0}(a)\,\eta_0(v)\colon a\in\mathcal{T}(V), v\in V\}$

— an even, hermitian linear functional ψ on V.

Given a generator Γ on V we denote by F_t^Γ the random variable

$$F_t^\Gamma(v)=a_t^\dagger(\eta_0(v))+\lambda_t(\rho_0(v))+a_t(\eta_0(v^*))+\psi_0(v)\,t\,\mathrm{id}$$

on $(\mathcal{S}(L^2(\mathbb{R}_+, D)), \Omega)$.

PROPOSITION 3.1. *Let Γ be a generator over V. Then $\mathcal{F}^\Gamma=(F_t^\Gamma)_{t\in\mathbb{R}_+}$ is a quantum stochastic process with independent and stationary increments on $(\mathcal{S}(L^2(\mathbb{R}_+)\otimes D), \Omega)$.*

Proof. For $s, t\in\mathbb{R}_+\cup\{\infty\}$, $s\leqslant t$, we write D_s^t for $L^2((s,t))\otimes D$. Let $t_1, ..., t_{n+1}\in\mathbb{R}_+$ be such that $t_1<\cdots<t_{n+1}$. Using (3.1), the natural decomposition

$$D_0^\infty\cong D_0^{t_1}\oplus D_{t_1}^{t_2}\oplus\cdots\oplus D_{t_n}^{t_{n+1}}\oplus D_{t_{n+1}}^\infty$$

gives

$$\mathcal{S}(D_0^\infty)\cong\mathcal{S}(D_0^{t_1})\otimes\mathcal{S}(D_{t_1}^{t_2})\otimes\cdots\otimes\mathcal{S}(D_{t_n}^{t_{n+1}})\otimes\mathcal{S}(D_{t_{n+1}}^\infty).$$

According to (3.2), we can decompose $a_t^\dagger(\eta_0(v))$ into

$$\begin{aligned}&a^\dagger(\chi_{(0,t_1)}\eta_0(v))\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}\\&\quad+\mathrm{id}\otimes a^\dagger(\chi_{(t_1,t_2)}\eta_0(v))\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}\\&\quad+\cdots+\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes a^\dagger(\chi_{(t_n,t_{n+1})}\eta_0(v))\otimes\mathrm{id}\\&\quad+\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes a^\dagger(\chi_{(t_{n+1},\infty)}\eta_0(v)).\end{aligned}$$

In the same manner, now using (3.3) and (3.4), we can decompose $a_t(\eta_0(v))$ and $\lambda_t(\rho(v))$. The commutation relations (3.5)–(3.9) yield

$$[F_{t_k,t_{k+1}}(v), F_{t_l,t_{l+1}}(w)]=0$$

for $k \neq l$ and $v, w \in V$. Since the vacuum Ω in $\mathscr{S}(D_0^\infty)$ is identified with the tensor product

$$\Omega_0^{t_1} \otimes \Omega_{t_2}^{t_1} \otimes \cdots \otimes \Omega_{t_n}^{t_{n+1}} \otimes \Omega_{t_{n+1}}^{\infty},$$

where Ω_s^t denotes the vacuum in $\mathscr{S}(D_s^t)$, independence of increments follows. Stationarity of increments follows from the fact that $\chi_{(0,t)}\eta_0(v) \in D_0^\infty$ and $\chi_{(0,t)}\rho_0(v) \in \mathbf{H}(D_0^\infty)$ are constant functions on the interval $[0, t]$. If φ_t is the distribution of F_t^Γ then for $v_1, ..., v_m \in V$ the function

$$t \mapsto \varphi_t(v_1 \cdots v_m) = \langle \Omega, F_t^\Gamma(v_1) \cdots F_t^\Gamma \Omega \rangle$$

is a polynomial vanishing at $t=0$ which yields $\varphi_t \to \delta$ in distribution for $t \downarrow 0$. ∎

Two generators $\Gamma_i = (D_i, \eta_{i,0}, \rho_{i,0}, \psi_{i,0})$, $i = 1, 2$, are called *equivalent* if there is a unitary operator $\mathscr{U}$ from the completion of D_1 to the completion of D_2 such that

$$\mathscr{U}\eta_{1,0}(v) = \eta_{2,0}(v)$$
$$\mathscr{U}\rho_{1,0}(v) = \rho_{2,0}(v)\mathscr{U}$$
$$\psi_{1,0}(v) = \psi_{2,0}(v).$$

for all $v \in V$. The term "generator over V" is motivated by the following proposition.

PROPOSITION 3.2. *Let $\Gamma = (D, \eta_0, \rho_0, \psi_0)$ be a generator over V. Then*

$$\psi^\Gamma(v_1 \cdots v_m) = \begin{cases} \psi_0(v_1) & \text{for } m=1 \\ \langle \eta_0(v_1^*), \eta_0(v_2) \rangle & \text{for } m=2 \\ \langle \eta_0(v_1^*), \rho_0(v_2) \cdots \rho_0(v_{m-1})\, \eta_0(v_m) \rangle & \text{for } m \geqslant 3 \end{cases}$$

defines an even, hermitian, conditionally positive linear functional on $\mathscr{T}(V)$. Two generators Γ_1 and Γ_2 are equivalent if and only if $\psi^{\Gamma_1} = \psi^{\Gamma_2}$. Conversely, for a given even, hermitian, conditionally positive linear functional ψ on $\mathscr{T}(V)$ there is a generator Γ over V such that $\psi^\Gamma = \psi$.

Proof. Let Γ be a generator over V. We denote by ρ the $*$-representation j_{ρ_0} of $\mathscr{T}(V)$ on D and define the linear mapping $\eta: \mathscr{T}(V) \to D$ by $\eta(\mathbf{1}) = 0$ and $\eta(av) = \rho(a)\, \eta_0(v)$ for $a \in \mathscr{T}(V)$, $v \in V$. We have for $a, b \in \operatorname{Kern} \delta$

$$\psi^\Gamma(ab) = \langle \eta(a^*), \eta(b) \rangle$$

which shows that ψ^Γ is conditionally positive. As ψ^Γ is positive on the $*$-algebra Kern δ, it must be hermitian on Kern δ. Since ψ_0 is hermitian it follows that ψ^{Γ} is hermitian. Clearly, ψ^{Γ} is even. One checks that ψ^{Γ} does not depend on the representative of the equivalence class of Γ.

Now let ψ be even, hermitian, and conditionally positive. The associated generator Γ is constructed as follows. Since ψ is conditionally positive and even, the sesquilinear form L_ψ on $\mathscr{T}(V)$ with

$$L_\psi(a, b) = \psi((a - \delta(a)\mathbf{1}))^* (b - \delta(b)\mathbf{1}))$$

$a, b \in \mathscr{T}(V)$, is positive and even. We devide $\mathscr{T}(V)$ by the null space $\mathscr{N}_\psi = \{a \in \mathscr{T}(V) \colon L_\psi(a, a) = 0\}$ of L_ψ to obtain the graded pre-Hilbert space $D = \mathscr{T}(V)/\mathscr{N}_\psi$. The canonical even, linear mapping from $\mathscr{T}(V)$ to D is denoted by η. Next we define the linear operators $\rho(a)$, $a \in \mathscr{T}(V)$, on D by

$$\begin{aligned} \rho(a)\,\eta(b) &= \eta(a(b - \delta(b)\mathbf{1})) \\ &= \eta(ab) - \eta(a)\,\delta(b), \end{aligned}$$

$b \in \mathscr{T}(V)$. We have

$$\begin{aligned} \|\rho(a)\,\eta(b)\|^2 &= \psi((b - \delta(b)\mathbf{1})^*\, a^* a(b - \delta(b)\mathbf{1})) \\ &= \langle \eta(b - \delta(b)\mathbf{1}), \eta(a^* a(b - \delta(b)\mathbf{1}))\rangle \\ &\leqslant \|\eta(b)\|\ \|\eta(a^* a(b - \delta(b)\mathbf{1}))\| \end{aligned}$$

by Cauchy–Schwartz inequality which shows that $\rho(a)$ is actually well defined. Moreover, ρ is an even *-representation of $\mathscr{T}(V)$. The quadruple $\Gamma = (D, \eta_0, \rho_0, \psi_0)$ with η_0, ρ_0, and ψ_0 the restrictions to V of η, ρ, and ψ, respectively, is a generator over V and $\psi = \psi^{\Gamma}$. It remains to be shown that $\psi^{\Gamma_1} = \psi^{\Gamma_2}$ implies $\Gamma_1 \sim \Gamma_2$. But if $\psi^{\Gamma_1} = \psi^{\Gamma_2}$ then the mapping

$$\mathscr{U} \colon D_1 \to D_2$$

with

$$\mathscr{U}\eta_1(b) = \eta_2(b),$$

$b \in \mathscr{T}(V)$, extends to a unitary operator from the completion of D_1 to the completion of D_2, and $\Gamma_1 \sim \Gamma_2$ follows. ▌

Remark. It follows from the proof of Proposition 3.2 that there is a 1–1 correspondence between generators over V and even, hermitian, conditionally positive linear functionals on $\mathscr{T}(V)$.

Now we come to the representation theorem.

THEOREM 3.3. *Let $\mathscr{F}$ be a quantum stochastic process with independent and stationary increments over V. Then there exists a generator Γ over V such that $\mathscr{F}^{\Gamma}$ and $\mathscr{F}$ are equivalent.*

Proof. Let ψ be the generator of the process $\mathscr{F}$ and let Γ be the generator over V associated to ψ by Proposition 3.2. If we show that the generator of $\mathscr{F}^\Gamma$ is also ψ the proof of the representation theorem is complete. Denote by $\tilde{\varphi}_t$ the distribution of F_t^Γ and by $\tilde{\psi}$ the generator of $\mathscr{F}^\Gamma$, i.e.,

$$\tilde{\psi}(a) = \frac{d}{dt}\tilde{\varphi}_t(a)\Big|_{t=0}.$$

For $m \in \mathbb{N}$, $v_1, ..., v_m \in V$, the function

$$t \mapsto \tilde{\varphi}_t(v_1 \cdots v_m) = \langle \Omega, F_t^\Gamma(v_1) \cdots F_t^\Gamma(v_m)\Omega \rangle \tag{3.13}$$

is a polynomial of degree m. We are interested in the linear term of this polynomial. For $m = 1$,

$$\begin{aligned}\tilde{\varphi}_t(v_1) &= \langle \Omega, (a_t^\dagger(\eta(v_1)) + \lambda_t(\rho(v_1)) + a_t(\eta(v_1)) + \psi(v_1)t)\Omega \rangle\\ &= \psi(v_1)t,\end{aligned}$$

because annihilation and preservation operators vanish on Ω. It follows that $\tilde{\psi}(v_1) = \psi(v_1)$. For $m = 2$,

$$\begin{aligned}\tilde{\varphi}_t(v_1 v_2) &= \langle \Omega, a_t(\eta(v_1))\, a_t^\dagger(\eta(v_2))\Omega \rangle + \psi(v_1)\,\psi(v_2)t^2\\ &= \langle \eta(v_1), \eta(v_2)\rangle t + \psi(v_1)\,\psi(v_2)t^2,\end{aligned}$$

and, therefore,

$$\begin{aligned}\tilde{\psi}(v_1 v_2) &= \langle \eta(v_1), \eta(v_2)\rangle\\ &= \psi(v_1 v_2).\end{aligned}$$

Similarly, the linear term of (3.13) for $m \geqslant 3$ is equal to

$$\begin{aligned}&\langle \Omega, a_t(\eta(v_1))\,\lambda_t(\rho(v_2)) \cdots \lambda_t(\rho(v_{m-1}))\, a_t^\dagger(\eta(v_m))\Omega \rangle\\ &\quad = \langle \eta(v_1), \eta(v_2 \cdots v_m)\rangle t\\ &\quad = \psi(v_1 \cdots v_m)t\end{aligned}$$

which proves $\tilde{\psi} = \psi$. ∎

4. Quantum Wiener and Poisson Processes

The simplest examples are the scalar processes. Let α be an even, hermitian linear functional on V. Define the linear functional d_α on $\mathscr{T}(V)$ by requiring it to vanish on all monomials of degree $\neq 1$ and to agree with α on V. Clearly, d_α is even, hermitian, and conditionally positive. The sesquilinear form L_{d_α} is 0 which means $D=\{0\}$ and $\mathscr{S}(L^2(\mathbb{R}_+)\otimes D)=\mathbb{C}$. Since $\eta=0$ and $\rho=0$, we have for the associated process $\mathscr{F}=(F_t)_{t\in\mathbb{R}_+}$

$$F_t(v)=\alpha(v)t\,\mathrm{id}.$$

It is clear that one can always *centralize* a quantum stochastic process with independent and stationary increments and that this is done on the level of the generator ψ by subtracting the linear functional d_α with $\alpha=\psi\restriction V$.

The following proposition yields a class of examples of centralized processes.

PROPOSITION 4.1. *Let Q be an even, positive sesquilinear form on V and let σ be an even state on the $*$-algebra $\mathscr{T}(V)$. Define the linear functional $\psi_{Q,\sigma}$ on $\mathscr{T}(V)$ by linear extension of*

$$\psi_{Q,\sigma}(\mathbf{1})=0$$

$$\psi_{Q,\sigma}(v)=0$$

$$\psi_{Q,\sigma}(vaw)=Q(v^*,w)\,\sigma(a),$$

$v, w\in V$, $a\in\mathscr{T}(V)$. Then $\psi_{Q,\sigma}$ is even, hermitian, and conditionally positive. The triplet (D,η,ρ) associated to $\psi_{Q,\sigma}$ by Proposition 3.2 and its proof is given by

$$D=E_Q\otimes E_\sigma$$

$$\eta(v)=\vartheta_Q(v)\otimes\Phi_\sigma$$

$$\rho_0(v)=\mathrm{id}\otimes\pi_\sigma(v),$$

where $E_Q=V/\mathscr{N}_Q$ with $\mathscr{N}_Q$ the null space of Q, ϑ_Q is the canonical mapping from V to E_Q and $(E_\sigma,\Phi_\sigma,\pi_\sigma)$ is the GNS-triplet associated to the state σ on the $*$-algebra $\mathscr{T}(V)$.

Proof. Clearly, $\psi_{Q,\sigma}$ is even and hermitian. An element a of Kern δ can always be written in the form

$$a=\sum_{i=1}^{m} a_i v_i$$

with $m\in\mathbb{N}$, $v_1,...,v_m\in V$, $a_1,...,a_m\in\mathscr{T}(V)$. We have

$$\begin{aligned}\psi_{Q,\sigma}(a^*a)&=\sum_{k,l=1}^{m}\psi_{Q,\sigma}(v_k^*a_k^*a_lv_l)\\&=\sum_{k,l=1}^{m}Q(v_k,v_l)\,\sigma(a_k^*a_l)\\&\geqslant 0,\end{aligned}$$

because the Schur product of two positive definite matrices is positive definite. This proves that $\psi_{Q,\sigma}$ is conditionally positive. We have for $c, d \in \mathbb{C}$, $a, b \in \mathscr{T}(V)$, $v, w \in V$

$$\begin{aligned} L_{\psi_{Q,\sigma}}(c\mathbf{1} + av, d\mathbf{1} + bw) &= \psi_{Q,\sigma}((av)^* bw) \\ &= Q(v, w)\, \sigma(a^* b) \end{aligned}$$

which proves the second assertion of the proposition. ▮

The generator ψ_Q on $\mathbb{C}[d]$ of the corresponding process over $\mathbb{C}^d$ of a d-dimensional Wiener process with diffusion matrix Q and drift vector 0 is given by

$$\psi_Q(x_1^{l_1} \cdots x_d^{l_d}) = 0 \qquad \text{for } \sum_{i=1}^{d} l_i \neq 2$$

$$\psi_Q(x_k x_l) = Q_{kl}.$$

We have $\psi_Q = \psi_{Q,\delta}$. A non-commutative generalization is obtained if we forget about the property of the classical covariance matrix Q to be symmetric and allow Q to be an arbitrary even, positive complex sesquilinear form on the graded vector space V with conjugation. We apply Proposition 4.1 to the special case $\psi_Q = \psi_{Q,\delta}$. Since $(E_\delta, \Phi_\delta, \pi_\delta) = (\mathbb{C}, 1, \delta)$, we have $D = E_Q$, and $\eta_0 = \vartheta_Q$. Moreover, $\rho(v) = \mathrm{id} \otimes \delta(v) = 0$ which means that there is no preservation term, and we have for the associated process

$$F_t(v) = a_t^\dagger(\vartheta_Q(v)) + a_t(\vartheta_Q(v^*)).$$

For even ξ a process of the form $(a_t^\dagger + a_t)(\xi)$ can be identified with a one-dimensional Wiener process via the Segal duality transformation; see [13]. Similarly, for odd ξ it can be identified with the so-called Clifford process; see [6]. If we choose an even, hermitian basis $\{v_i : i \in I\}$ of V we have a family $(W_{t,i})_{i \in I}$ of Wiener and Clifford processes

$$W_{t,i} = (a_t^\dagger + a_t)(\vartheta_Q(v_i))$$

which do not commute if Q is not symmetric. (If Q is symmetric we have a multivariate Wiener and a multivariate Clifford process.) In particular, for $V = \mathbb{C}^2 = \mathrm{Lin}\{x_1, x_2\}$, $x_1^* = x_2$, and

$$Q = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},$$

we have $E_Q = \mathbb{C}$, $\eta(x_1) = 1/\sqrt{2}$, $\eta(x_2) = -i/\sqrt{2}$, so that

$$W_{t,1} = \frac{1}{\sqrt{2}}(a_t^\dagger + a_t)$$
$$W_{t,2} = \frac{-i}{\sqrt{2}}(a_t^\dagger - a_t).$$

This is the quantum Wiener process in the Bose case [7] if we choose the trivial graduation $\deg x_1 = \deg x_2 = 0$ on V, and in the Fermi case [4] if we choose $\deg x_1 = \deg x_2 = 1$. We call a process with independent and stationary increments and with generator of the form ψ_Q, $Q \neq 0$, *a quantum Wiener process.*

The generator of the corresponding process over $\mathbb{C}$ of a Poisson process with intensity $\beta > 0$ and with jumps of length $\alpha \in \mathbb{R}$ is equal to $\beta(\varepsilon_\alpha - \delta)$, where $\varepsilon_\alpha(x^n) = \alpha^n$, $n \in \mathbb{N} \cup \{0\}$. This is also of the form $\psi_{Q,\sigma}$. It is the special case where $V = \mathbb{C}$, the state σ is the $*$-algebra homomorphism with $\sigma(x) = \alpha$, and $Q = \beta\alpha^2$. We call the process associated to $\psi_{Q,\sigma}$ a *non-commutative Poisson process* if $Q \neq 0$ and if $\sigma \neq \delta$ is a $*$-algebra homomorphism. This is motivated by the following considerations. Let the graduation of V be trivial. Choose a hermitian basis $\{v_i\}$ of V such that $\sigma(v_i) = \alpha_i \neq 0$. Assuming also that V is finite-dimensional, $\dim V = d$, we can write Q as the sum of positive definite matrices of rank 1. Thus $\psi_{Q,\sigma}$ is a sum of linear functionals $\psi_{\beta,\alpha}$, where $\beta = (\beta_1, ..., \beta_d) \in \mathbb{C}^d$ and $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{R}^d$, $\alpha_i \neq 0$, and

$$\psi_{\beta,\alpha}(v_{i_1} \cdots v_{i_m}) = \begin{cases} 0 & \text{for} \quad m = 1 \\ \bar{\beta}_{i_1}\beta_{i_2} & \text{for} \quad m = 2 \\ \bar{\beta}_{i_1}\beta_{i_m}\alpha_{i_2} \cdots \alpha_{i_{m-1}} & \text{for} \quad m \geqslant 3. \end{cases}$$

The triplet (D, η, ρ) associated to $\psi_{\beta,\alpha}$ is given by $D = \mathbb{C}$, $\eta(v_i) = \beta_i$, and $\rho(v_i) = \alpha_i$. The associated process is given by

$$F_t(v_i) = \beta_i a_t^\dagger + \bar{\beta}_i a_t + \alpha_i \lambda_t.$$

For each $i \in \{1, ..., d\}$ the restriction of this process to $\operatorname{Lin}\{v_i\}$ is a centralized Poisson process of intensity $|\beta_i|^2/\alpha_i^2$ and with jumps of length α_i; see [15, 25]. But in general the Poisson processes $F_t(v_k)$ and $F_t(v_l)$, $k \neq l$, do not commute.

Finally, we show that it is always possible to extract "the maximal quantum Wiener component" from a given process with independent and stationary increments.

Two quantum stochastic processes $\mathscr{F}_1$ and $\mathscr{F}_2$ on the same space $(\mathscr{D}, \Phi)$ are said to be *independent* if the algebras generated by $\{F_{1,t}(v) \colon t \in \mathbb{R}_+, v \in V\}$ and $\{F_{2,t}(v)\colon t \in \mathbb{R}_+, v \in V\}$ are independent. A quantum stochastic process $\mathscr{F}$ is the *sum* of two quantum stochastic processes $\mathscr{F}_1$ and $\mathscr{F}_2$, in symbols $\mathscr{F} = \mathscr{F}_1 \oplus \mathscr{F}_2$, if $\mathscr{F}$ is stochastically equivalent to the process $(F_{1,t} + F_{2,t})_{t \in \mathbb{R}_+}$ and if $\mathscr{F}_1$ and $\mathscr{F}_2$ are independent. If $\mathscr{F} = \mathscr{F}_1 \oplus \mathscr{F}_2$ for $\mathscr{F}$, $\mathscr{F}_1$, and $\mathscr{F}_2$ processes with independent and stationary increments, then

$$\varphi_t = \varphi_{1,t} * \varphi_{2,t}$$

which implies

$$\psi = \psi_1 + \psi_2.$$

Conversely, let ψ, ψ_1, and ψ_2 be even, hermitian, conditionally positive linear functionals on $\mathscr{T}(V)$ such that $\psi = \psi_1 + \psi_2$. Then $\mathscr{F}^\Gamma$ is the sum of the processes $(F_t^{\Gamma_1} \otimes \mathrm{id})_{t \in \mathbb{R}_+}$ and $(\mathrm{id} \otimes F_t^{\Gamma_2})_{t \in \mathbb{R}_+}$ on $(\mathscr{S}(L^2(\mathbb{R}_+) \otimes D_1) \otimes \mathscr{S}(L^2(\mathbb{R}_+) \otimes D_2), \Omega_1 \otimes \Omega_2)$, which is, by the isomorphism (3.1), equal to the process

$$\begin{aligned}(a_t^\dagger(\eta_1(v)) + \lambda_t(\rho_1(v)) + a_t(\eta_1(v^*)) + \psi_1(v)t\,\mathrm{id})\\ + (a_t^\dagger(\eta_2(v)) + \lambda_t(\rho_2(v)) + a_t(\eta_2(v^*)) + \psi_2(v)t\,\mathrm{id})\end{aligned}$$

on $(\mathscr{S}(L^2(\mathbb{R}_+) \otimes (D_1 \oplus D_2)), \Omega)$. A *quantum Wiener component* of a process $\mathscr{F}$ with independent and stationary increments is a quantum Wiener process $\mathscr{F}^Q$, such that $\mathscr{F} = \mathscr{F}^Q \oplus \mathscr{R}$, where $\mathscr{R}$ is another process with independent and stationary increments. A quantum Wiener component is called *maximal* if $\mathscr{R}$ has no quantum Wiener component.

THEOREM 4.2. *Let $\mathscr{F}$ be a quantum stochastic process with independent and stationary increments and let ψ be its generator. If*

$$D^2 = \{\eta(ab) : a, b \in \mathrm{Kern}\,\delta\}$$

is not total in the completion H of D then $\mathscr{F}$ has a (up to stochastic equivalence) uniquely determined maximal quantum Wiener component $\mathscr{F}^Q$. The sequilinear form Q is given by

$$Q(v, w) = \langle (\mathrm{id} - \mathscr{P})\,\eta(v^*), \eta(w) \rangle, \tag{4.1}$$

where $\mathscr{P}$ denotes the projection onto the closure of the linear span of D^2 in H. If D^2 is total in H then $\mathscr{F}$ has no quantum Wiener component.

Proof. Our considerations above show that everything can be reduced to the level of the generators. It was shown in [3] that for Q defined by (4.1) $\psi - \psi_Q$ is conditionally positive and that "$\psi - \psi_P$ conditionally positive for $P \geqslant 0$" implies $P \leqslant Q$. This gives the maximality of $\mathscr{F}^Q$ if $Q \neq 0$.

If $Q=0$ we have that "$\psi-\psi_P$ conditionally positive" implies $P=0$ and $\mathscr{F}$ has no quantum Wiener component. This completes the proof. ∎

It follows from Theorem 4.2 that a process with generator $\psi_{Q,\sigma}$ has no quantum Wiener component if and only if $\{\pi_\sigma(a)\Phi_\sigma\colon a \in \operatorname{Kern}\delta\}$ is total in the completion of E_σ. Clearly, the latter is true if σ is an algebra homomorphism and $\sigma \neq \delta$. Thus we have that *a quantum Poisson process has no quantum Wiener component.*

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